A SHORT DEMONSTRATION OF THE EXPONENTIAL THEOREM.

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To demonstrate the Exponential Theorem without the aid of the calculus, various methods have been devised by mathematicians; but all of the methods known to the writer are much longer than the one here given, which, for brevity and directness of proof, leaves but little to be desired.

Assuming that the equation

$$a^x = c_0 + c_1x + c_2x^2 + c_3x^3 + c_4x^4 + \dots \quad (1)$$

is true for all values of x, then c_0 must equal unity. Squaring (1) we have

$$a^{2x} = 1 + 2c_1x + (2c_2 + c_1^2)x^2 + (2c_3 + 2c_2c_1)x^3 + (2c_4 + 2c_3c_1 + c_2^2)x^4 + \dots, \quad (2)$$

while, if in equation (1) we substitute $2x$ in place of x, we have

$$a^{2x} = 1 + 2c_1x + 4c_2x^2 + 8c_3x^3 + 16c_4x^4 + \dots. \quad (3)$$

Equating the coefficients of the like powers of x in (2) and (3), we at once obtain

$$c_1 = c_1; \quad c_2 = \frac{c_1^2}{2!}; \quad c_3 = \frac{c_1^3}{3!}; \quad c_4 = \frac{c_1^4}{4!}; \text{ etc.};$$

and if the succeeding values of c follow the same law,

$$c_{n-1} = \frac{c_1^{n-1}}{(n-1)!}, \quad c_n = \frac{c_1^n}{n!},$$

which is readily proved by equating the coefficients of the nth power of x in the second members of (2) and (3). Thus

$$2c_n + 2c_{n-1}c_1 + 2c_{n-2}c_2 + 2c_{n-3}c_3 + \dots + 2c_{\frac{1}{2}(n+1)}c_{\frac{1}{2}(n-1)} \left(\text{or } c_{\frac{1}{2}n}^2\right) = 2^nc_n, \quad (4)$$

which, if we substitute the assumed values of c_n, c_{n-1}, c_{n-2}, etc., becomes

$$\frac{c_1^n}{n!}\left[2 + 2\binom{n}{1} + 2\binom{n}{2} + 2\binom{n}{3} + \dots + 2\binom{n}{\frac{1}{2}(n-1)} \text{ or } \binom{n}{\frac{1}{2}n}\right] = 2^n\frac{c_1^n}{n!}. \quad (5)$$

When n is even, the last term within the [ ] is $\binom{n}{\frac{1}{2}n}$, and when n is odd, the last term is $2\binom{n}{\frac{1}{2}(n-1)}$. Now, as the value of the expression within the [ ] in equation (5) is always equal to 2^n, it follows that, in general, c_n must always equal $\frac{c_1^n}{n!}$; and hence

$$a^x = 1 + c_1x + \frac{c_1^2x^2}{2!} + \frac{c_1^3x^3}{3!} + \frac{c_1^4x^4}{4!} + \dots + \frac{c_1^nx^n}{n!} + \dots.$$